AiMT Advances in Military Technology
Vol. 5, No. 2, December 2010

Modelling and Control of a Quadrocopter

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The manuscript was received on 31 May 2010 and was accepted after revision for publication on 14 September 2010.

Abstract:

The paper describes model and development of full control of a quadrotor aerial robot. The mathematical model of quadrotor is nonlinear system based on Newton law of motion of rigid body. Control system of the quadrotor is designed with help of state variable approach and also with help of corresponding physical approach which was inspired by the former one. The quality of the model and its control is tested by simulation and on a real flying model as well.

Keywords:

Quadrocopter, quadrotor, control, flying robot, VTOL aircraft, robot

1. Introduction

Quadrocopter or quadrotor is VTOL (Vertical Take-Off and Landing) aerial vehicle belonging to the class of multirotor helicopters. They differ from the standard helicopters in using rotors with fixed-pitch blades, thus their rotor pitch does not vary as the blades rotate. Quadrotor uses 4 rotors, but one can meet constructions with 6 or 8 rotors. Perhaps the first multirotor helicopter with fixed pitch blades rotors appeared in 1923 (De Bothezat), but technology at that time was not ready for applicable construction of such a machine. The main problem was inherent instability of the vehicle and thus pilot workload which was too high. Standard helicopters though more mechanically complex [1] proved to be more applicable. Later progress in technology removed the most serious obstacles in controllability of quadrotor type aircraft. Thus the idea of construction of extremely simple helicopter appeared again, this time in hobby industry. First commercially available hobbyist quadrotor Draganflyer appeared around 2000 [2]. Since then the development of light batteries, brushless actuators and MEMS sensors allowed considerable improvement in construction and control of this

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type of helicopter. Today's quadrocopters are not only radio-controlled toys but they are robust and simple helicopters that are able to bear nearly two kilograms payload [3]. They have started to be used as UAV (Unmanned Aerial Vehicles) or aerial robots mainly for reconnaissance purposes. Military applications are ready to come soon.

Apart from technological issues, quadrocopter modelling and control studies are still attractive themes for researchers. Lots of control strategies are studied in great number of articles starting with standard methods using PID control technique [4] up to complicated nonlinear techniques [5, 6] or techniques of softcomputing [7].

This article describes a simple method of control that can be easily implemented on microprocessor and it has proved to be applicable for full control of light indoor quadrotor.

2. Mathematical Model of Quadrocopter

Mathematical model of the quadrotor can be derived according to Newton's laws of motion in the same manner as for ordinary aircraft [8]. Kinematic scheme of the quadrotor is depicted in Fig. 1.

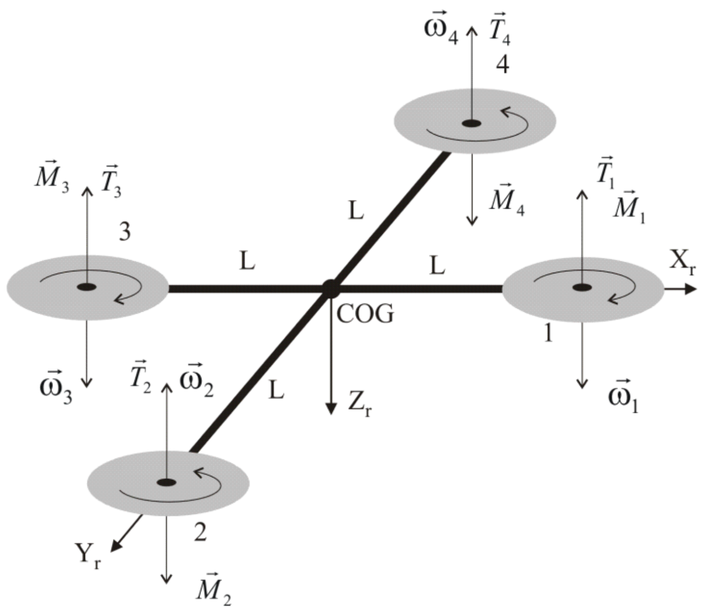

Fig. 1 Kinematic scheme of the quadrotor

Origin of reference frame $(X,Y,Z)_r$ that is firmly coupled with quadrotor is placed in its centre of gravity. Fig. 1 also shows positive directions of rotation ω_i, resp. positive direction of thrusts T_i, resp. positive direction of reactive torques M_i of individual propellers.

Newton's laws of motion are written in vector form [8],

$$\sum \vec{\boldsymbol{F}} = \frac{\mathrm{d}}{\mathrm{dt}}(m\vec{\boldsymbol{V}})\bigg|_{0}, \qquad (1)$$

$$\sum \vec{\boldsymbol{M}} = \left.\frac{\mathrm{d}\vec{\boldsymbol{H}}}{\mathrm{dt}}\right|_{0}, \tag{2}$$

where the left-hand sides represent the sum of all forces resp. all moments acting on the quadrotor's body. Subscript *0* denotes time derivative of momentum ($m\vec{\boldsymbol{V}}$) and moment of momentum ($\vec{\boldsymbol{H}}$) with respect to inertial frame.

Orientation of quadrotor reference frame with respect to inertial frame – Earth $(X,Y,Z)_0$ is depicted in Fig. 2.

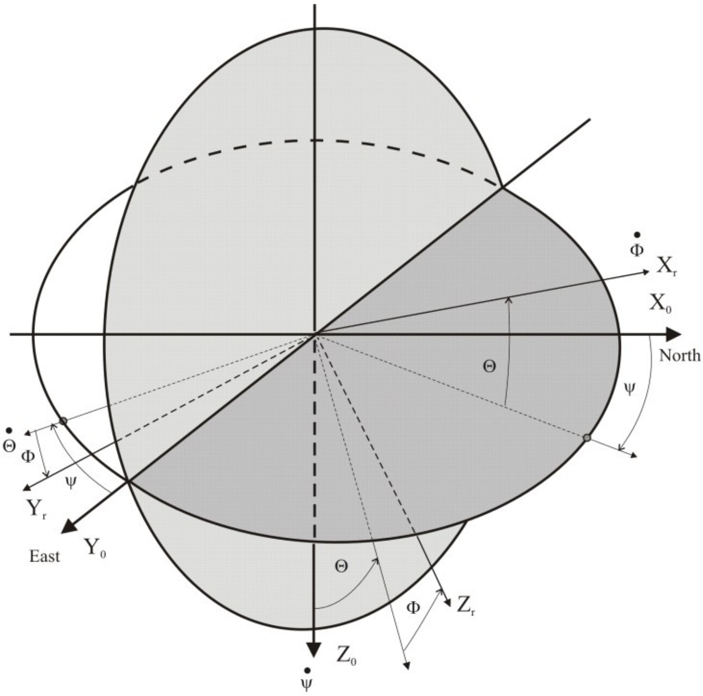

Fig. 2 Orientation of frames

Mutual orientation of frames is described by matrix of rotation $\boldsymbol{R}_{0\mathrm{r}}$ [9],

$$\boldsymbol{R}_{0r} = \begin{bmatrix} c\Psi c\Theta & -s\Psi c\Phi + c\Psi s\Theta s\Phi & -s\Psi s\Phi + c\Psi s\Theta c\Phi \\ s\Psi c\Theta & c\Psi c\Phi + s\Psi s\Theta s\Phi & -c\Psi s\Phi + s\Psi s\Theta c\Phi \\ -s\Theta & c\Theta s\Phi & c\Theta c\Phi \end{bmatrix}, \tag{3}$$

where c stands for *cos* and s stands for *sin*. Angles Φ, Θ, Ψ are angles of roll, pitch and yaw.

Equation (1) can be conveniently expressed in quadrotor reference frame

$$\begin{aligned} \sum F_{xr} &= m(\dot{v}_{xr} + v_{zr}\omega_{yr} - v_{yr}\omega_{zr}), \\ \sum F_{yr} &= m(\dot{v}_{yr} + v_{xr}\omega_{zr} - v_{zr}\omega_{xr}), \\ \sum F_{zr} &= m(\dot{v}_{zr} + v_{yr}\omega_{xr} - v_{xr}\omega_{yr}). \end{aligned} \tag{4}$$

Left-hand sides (4) represent sums of components of all forces acting on quadrotor in individual axis. v_{ir}, ω_{ir} are components of velocity and angular velocity in respective axis i.

Equation (2) can be expressed in quadrotor reference frame as well, but one must know the distribution of mass in the quadrotor body. One can realistically suppose that the mass of the quadrotor is distributed with respect to planes $(XY)_r$ $(XZ)_r$ and $(YZ)_r$ symmetrically. Then the products of inertia can be neglected and (2) can be written as follows

$$\begin{aligned}\sum M_{xr} &= J_x\dot{\omega}_{xr} + \omega_{yr}\omega_{zr}(J_z - J_y),\\ \sum M_{yr} &= J_y\dot{\omega}_{yr} + \omega_{zr}\omega_{xr}(J_x - J_z),\\ \sum M_{zr} &= J_z\dot{\omega}_{zr} + \omega_{xr}\omega_{yr}(J_y - J_x).\end{aligned} \tag{5}$$

Left-hand sides (5) represent sums of components of all moments acting on quadrotor in respective axis i. J_i are moments of inertia with respect to respective axis i. Due to symmetry of quadrotor one can also assume that $J_x = J_y$.

Now one must express motion of quadrotor in inertial frame $(X,Y,Z)_0$, see Fig. 2. Well known relation between angular velocity of a body in inertial and in aircraft frame is given by the projection of pitch roll and yaw rate in aircraft frame and vice versa [8].

$$\begin{aligned}\dot{\Theta} &= \omega_{yr}\cos\Phi - \omega_{zr}\cos\Phi,\\ \dot{\Phi} &= \omega_{zr} + \omega_{yr}\sin\Phi\tan\Theta + \omega_{zr}\cos\Phi\,\tan\Theta,\\ \dot{\Psi} &= \omega_{yr}\frac{\sin\Phi}{\cos\Theta} + \omega_{zr}\frac{\cos\Phi}{\cos\Theta}.\end{aligned} \tag{6}$$

Relation between velocity in inertial and aircraft frame can be expressed in the following form.

$$\begin{bmatrix} v_{x0}\\ v_{y0}\\ v_{z0}\end{bmatrix} = \boldsymbol{R}_{0r}\begin{bmatrix} v_{xr}\\ v_{yr}\\ v_{zr}\end{bmatrix}. \tag{7}$$

Finally forces and moments acting on quadrotor must be specified, which is the most complicated problem. Only slow motions of hovering quadrotor will be supposed further on. Then the main forces acting on quadrotor are thrusts of propellers and gravity. Thus

$$\begin{aligned}\sum F_{xr} &= -mg\,\sin\Theta,\\ \sum F_{yr} &= mg\,\cos\Theta\sin\Phi,\\ \sum F_{zr} &= -T_1 - T_2 - T_3 - T_4 + mg\,\cos\Theta\cos\Phi,\end{aligned} \tag{8}$$

where T_i stands for thrust of ith propeller.

Main moments are reactive and gyroscopic moments of propellers and moments due to thrust of propellers.

$$\begin{aligned}
\sum M_{xr} &= -LT_2 + LT_4 + J_{mp}\omega_{yr}(\omega_1 - \omega_2 + \omega_3 - \omega_4),\\
\sum M_{yr} &= LT_1 - LT_3 + J_{mp}\omega_{xr}(-\omega_1 + \omega_2 - \omega_3 + \omega_4),\\
\sum M_{zr} &= k_{MT}(-T_1 + T_2 - T_3 + T_4).
\end{aligned} \tag{9}$$

where L is the length of arm of quadrotor, see Fig. 1. J_{mp} is moment of inertia of a motor with propeller and ω_i represents angular velocity of individual propellers. Reactive moments of propellers are assumed to be proportional to thrust of propeller, thus k_{MT} represents the ratio of reactive moment and thrust of used propellers. Similarly angular velocities of propellers are assumed to be proportional to thrusts of propellers, $\omega_i = k_{\omega T}T_i$. Equations (4)-(9) are used for simulation of flight of quadrotor and for testing of control law. They can be transformed to state variable form as follows,

$$\begin{aligned}
\dot{v}_{xr} &= -v_{zr}\omega_{yr} + v_{yr}\omega_{zr} - gs\Theta,\\
\dot{v}_{yr} &= -v_{xr}\omega_{zr} + v_{zr}\omega_{xr} + gc\Theta s\Phi,\\
\dot{v}_{zr} &= -v_{yr}\omega_{xr} + v_{xr}\omega_{yr} + gc\Theta c\Phi - \frac{T}{m},\\
\dot{\omega}_{xr} &= \frac{1}{J_x}\left[-\omega_{yr}\omega_{zr}(J_z - J_y) + M_x - \frac{k_{\omega T}}{k_{MT}}J_{mp}M_z\omega_{yr}\right],\\
\dot{\omega}_{yr} &= \frac{1}{J_y}\left[-\omega_{xr}\omega_{zr}(J_x - J_z) + M_y - \frac{k_{\omega T}}{k_{MT}}J_{mp}M_z\omega_{xr}\right],\\
\dot{\omega}_{zr} &= \frac{M_z}{J_z},\\
\dot{\Theta} &= \omega_{yr}c\Phi - \omega_{zr}s\Phi,\\
\dot{\Phi} &= \omega_{xr} + \omega_{yr}s\Phi\tan\Theta + \omega_{zr}c\Phi\tan\Theta,\\
\dot{\Psi} &= \omega_{yr}\frac{s\Phi}{c\Theta} + \omega_{zr}\frac{c\Phi}{c\Theta},\\
\dot{x} &= c\Psi c\Theta v_{xr} + (-s\Psi c\Phi + c\Psi s\Theta s\Phi)v_{yr} + (s\Psi s\Phi + c\Psi s\Theta c\Phi)v_{zr},\\
\dot{y} &= s\Psi c\Theta v_{xr} + (c\Psi c\Phi + s\Psi s\Theta s\Phi)v_{yr} + (-c\Psi s\Phi + s\Psi s\Theta c\Phi)v_{zr},\\
\dot{z} &= -s\Theta v_{xr} + c\Theta s\Phi v_{yr} + c\Theta c\Phi v_{zr},
\end{aligned} \tag{10}$$

where $[x,y,z]^T$ represents position of quadrotor in inertial frame $(X,Y,Z)_0$.

In (10) the natural manipulated variables (thrusts of propellers) are replaced by new ones (total thrust and moments). These variables are related by the following one-to-one relation

$$\begin{bmatrix} T \\ M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -L & 0 & L \\ L & 0 & -L & 0 \\ -k_{MT} & k_{MT} & k_{MT} & k_{MT} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}. \tag{11}$$

3. Control Law Design

The quadrotor is supposed to be used for reconnaissance missions mainly, then its prevailing state will be hovering in equilibrium or slow flight. All state variables are equal to zero during hovering in equilibrium. Substituting these zero values in (10) yields equilibrium values of manipulated variables

$$\begin{aligned} T_0 &= mg, \\ M_{x0} &= M_{y0} = M_{z0} = 0. \end{aligned} \tag{12}$$

Now one can linearize state equations (10) assuming small values of state variables and small differences of manipulated variables from equilibrium. Using standard linearization technique [10] yields standard equation of linear system

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}, \tag{13}$$

where

$$\begin{aligned} &\boldsymbol{x} = \begin{bmatrix} v_{xr} & v_{yr} & v_{zr} & \omega_{xr} & \omega_{yr} & \omega_{zr} & \Theta & \Phi & \Psi & x & y & z \end{bmatrix}^T, \\ &\boldsymbol{u} = \begin{bmatrix} \Delta T & \Delta M_x & \Delta M_y & \Delta M_z \end{bmatrix}^T, \\ &\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} & \boldsymbol{A}_{14} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} & \boldsymbol{A}_{24} \\ \boldsymbol{A}_{31} & \boldsymbol{A}_{32} & \boldsymbol{A}_{33} & \boldsymbol{A}_{34} \\ \boldsymbol{A}_{41} & \boldsymbol{A}_{42} & \boldsymbol{A}_{43} & \boldsymbol{A}_{44} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \\ \boldsymbol{B}_{31} & \boldsymbol{B}_{32} \\ \boldsymbol{B}_{41} & \boldsymbol{B}_{42} \end{bmatrix}, \\ &\boldsymbol{A}_{32} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{A}_{41} = \boldsymbol{I}, \boldsymbol{B}_{11} = \begin{bmatrix} 0 \\ 0 \\ -1/m \end{bmatrix}, \boldsymbol{B}_{22} = \begin{bmatrix} 1/J_x & 0 & 0 \\ 0 & 1/J_y & 0 \\ 0 & 0 & 1/J_z \end{bmatrix}. \end{aligned} \tag{14}$$

Other matrices are (3×3) $\boldsymbol{A}_{ij} = \boldsymbol{0}$, (3×1) $\boldsymbol{B}_{i1} = \boldsymbol{0}$, (3×3) $\boldsymbol{B}_{i2} = \boldsymbol{0}$.

Of course linear model is valid for low velocities and for small angles of orientation of quadrotor.

System (13) proved to be controllable by full state variable feedback

$$\boldsymbol{u} = \boldsymbol{Kx}\,. \tag{15}$$

Linearized state variable equations reveal simple structure of controlled plant that is displayed in Fig. 3. The structure shows also the fact that movement in water level plane is possible only because of pitching and banking of quadrotor. Because of small values of angles Θ, Φ and Ψ, there is small difference between corresponding velocities in quadrotor and fixed inertial frame.

Motion of controlled plant in altitude, rolling, pitching and yawing is completely decoupled and can be easily controlled; e.g. for pitch angle the simplified model of motion is

$$\ddot{\Theta} = \frac{1}{J_y} \Delta M_y\,. \tag{16}$$

This model can be controlled by simple PD controller in form

$$\Delta M_y = J_y(-K_{\omega y}\dot{\Theta} - K_{\Theta}\Theta + K_{\Theta}\Theta_r)\,, \tag{17}$$

where Θ_r is the reference value.

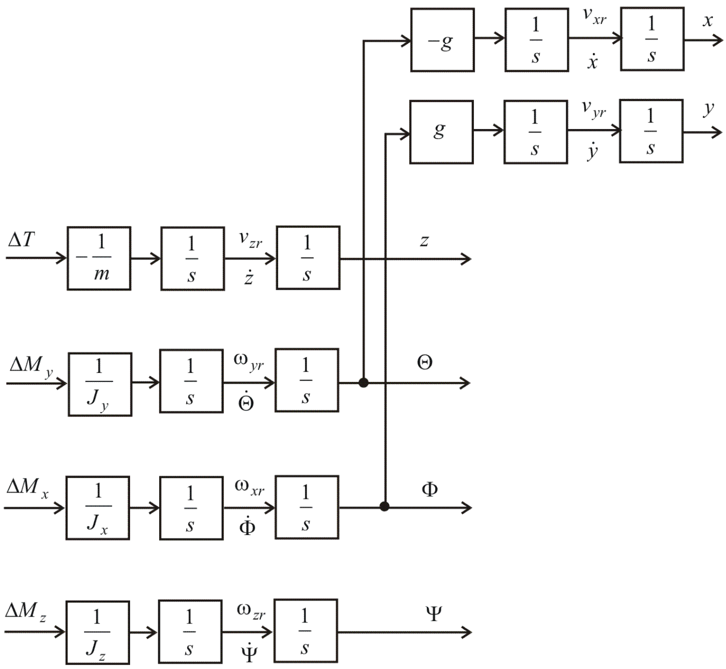

Fig. 3 Structure of linearized controlled plant

Combination of control law (17) with model (16) yields equation that describes pitch motion of quadrotor together with controller

$$\ddot{\Theta} + K_{\omega y}\dot{\Theta} + K_{\Theta}\Theta = K_{\Theta}\Theta_r \,, \tag{18}$$

Comparison of (18) with standard equation of second-order linear system (19)

$$\ddot{\Theta} + 2\xi\omega_n\dot{\Theta} + \omega_n^2\Theta = \omega_n^2\Theta_r \,, \tag{19}$$

yields rule for choice of controller parameters.

$$K_{\Theta} = \omega_n^2 \qquad K_{\omega y} = 2\xi\sqrt{K_{\Theta}} \,. \tag{20}$$

The higher ω_n, the faster will transient response of complete system be. $\xi \geq 0.5$ gives reasonable damping of the response. Because $\omega_{yr} \approx d\Theta/dt$ one can use as velocity feedback signal ω_{yr} that can be easily measured by MEMS rate gyro. Control law (17) will be then

$$\Delta M_y = J_y(-K_{\omega y}\omega_{yr} - K_{\Theta}\Theta + K_{\Theta}\Theta_r) \,. \tag{21}$$

Signal Θ can be measured by MEMS accelerometer (inclinometer).

Controllers of other angles of orientation and controller of altitude can be designed in the same way. Thus control law of controller – stabilizer of altitude and orientation only, will be

$$
\begin{aligned}
\Delta M_x &= J_x(-K_{\omega x}\omega_{xr} - K_\Phi \Phi + K_\Phi \Phi_r),\\
\Delta M_y &= J_y(-K_{\omega y}\omega_{yr} - K_\Theta \Theta + K_\Theta \Theta_r),\\
\Delta M_z &= J_z(-K_{\omega z}\omega_{zr} - K_\Psi \Psi + K_\Psi \Psi_r),\\
T &= mg + m(-K_{vz}\dot{z} - K_z z + K_z z_r),
\end{aligned} \tag{22}
$$

where usually will be $\Theta_r = \Phi_r = \Psi = 0$ and $z_r = \text{const}$.

Control of position (x and y) can be done via angles of pitch and roll. Achieving fast response of control of these values one can suppose that at any time it will be valid $\Theta(t) = \Theta_r(t)$ and $\Phi(t) = \Phi_r(t)$. Thus one can write (see Fig. 3.):

$$
\begin{aligned}
\ddot{x} &= -g\Theta_r,\\
\ddot{y} &= g\Phi_r.
\end{aligned} \tag{23}
$$

Similarly as in previous case, the controller of x and y can be designed in form

$$
\begin{aligned}
\Theta_r &= \frac{1}{g}(K_{vx}\dot{x} + K_x x - K_x x_r),\\
\Phi_r &= \frac{1}{g}(-K_{vy}\dot{y} - K_y y + K_y y_r).
\end{aligned} \tag{24}
$$

These control laws result in equations that describe x, y motion of quadrotor together with controller

$$
\begin{aligned}
\ddot{x} + K_{vx}\dot{x} + K_x x &= K_x x_r,\\
\ddot{y} + K_{vy}\dot{y} + K_y y &= K_y y_r,
\end{aligned} \tag{25}
$$

where x_r and y_r are reference values of position. Parameters of the controller can be calculated in the same way as in the previous case. Thus law of full control of quadrotor is given by equations (22) and (25).

The full control of quadrotor has been simulated with the following parameters $m = 0.8$ kg, $J_x = J_y = 1.8 \times 10^{-3}$ kgm^2, $J_z = 1.5 \times 10^{-3}$ kgm^2, $L = 0.2$ m, $k = 0.1$ m. Parameters for the controller of orientation and altitude were $\omega_n = 10$ s^{-1}, $\xi = 1$. Parameters for the controller of position x, y were $\omega_n = 1$ s^{-1}, $\xi = 1$. Maximum possible thrust of each propeller is 3.9 N.

Fig. 4 shows motion of quadrotor and forces of its propellers in response for ramp input of reference x_r. One can clearly see that the forward motion of the quadrotor is due to pitch. One can also observe how pitch was attained by change of thrust of individual propellers.

4. Conclusion

This article describes a relatively simple method of control of highly unstable helicopter – quadrotor. The main advantage of the proposed control law is in its clear physical interpretation that allows final tuning of control parameters during flight. The control law has been implemented in radio controlled micro quadrotor of approximate weight and size mentioned above. The proposed control law requires fast response of actuators (thrust) to actuating variable and fast processing of control algorithm. In our implementation there have been used fast and light brushless motors LAU 20-1300 (mass approx. 0.025 kg and power approx. 40 W) [11]. The control law was implemented in microcontroller ATMEGA644P. Because of the problems with sensors of positions *x*, *y*, and *z*, only the stabilizer of orientation *Θ*, *Φ* ,*Ψ* has been implemented. The control of position is still performed manually by human operator. We suppose to use the quadrotor as a flying robot in a swarm of mobile reconnaissance robots.

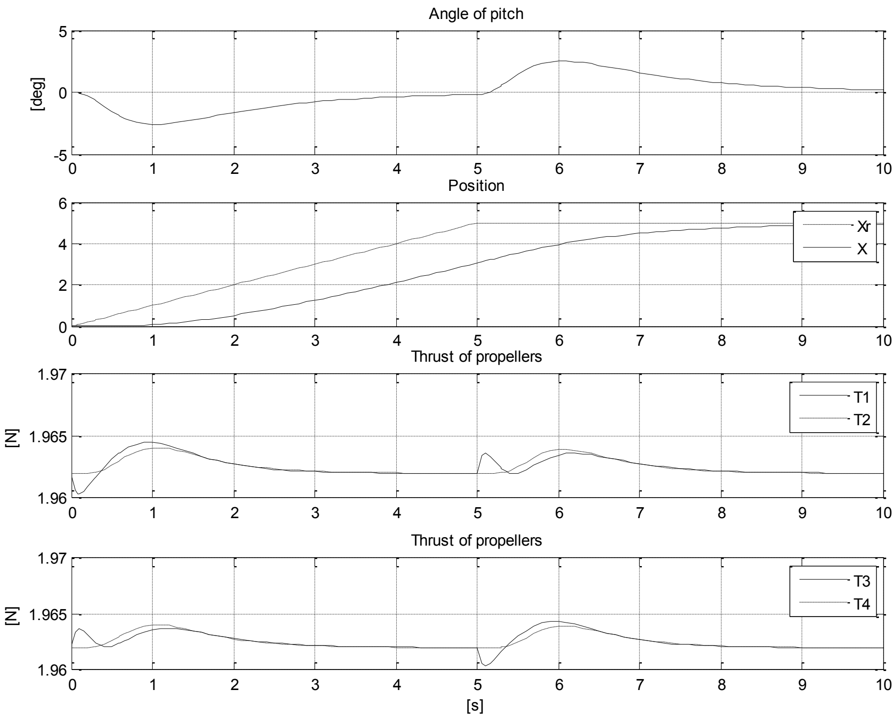

Fig. 4 Results of simulation

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Acknowledgement

The work presented in this paper has been supported by the by the Ministry of Education of the Czech Republic (research project No. 1M0567).